Homework 11 Solution

1. Sec. 6.4 Q7

7. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

(a) If T is self-adjoint, then T_W is self-adjoint.
(b) $W^\perp$ is T^*-invariant.
(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

$T: V \to V$ is linear, $W \subset V$ is T-invariant

(a) if $T = T^*$, $\quad \forall x, y \in W$.

$$\langle T_W(x), y \rangle = \langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

$$\langle (T_W)^*(x), y \rangle = \langle x, T_W(y) \rangle = \langle x, T(y) \rangle$$

So $T_W(x) = (T_W)^*(x) \quad \forall x \in W$. $\quad$ i.e. $T_W = (T_W)^*$

(b) $\forall y \in W^\perp$. $\forall x \in W$. we have $T(x) \in W$.

$$\langle T^*(y), x \rangle = \langle \underset{\substack{\cap \\ W^\perp}}{y}, \underset{\substack{\cap \\ W}}{T(x)} \rangle = 0$$

So $T^*(y) \in W^\perp$ i.e. $W^\perp$ is T^*-invariant

(c) $\forall x \in W$, $y \in W$.

$$\langle (T_W)^*(x), y \rangle = \langle x, T_W(y) \rangle = \langle x, T(y) \rangle = \langle (T^*)_W(x), y \rangle = \langle T^*(x), y \rangle$$

Thus $(T_W)^*(x) = (T^*)_W(x) \quad \forall x \in W$. $\quad$ i.e. $(T_W)^* = (T^*)_W$

(d) $\forall x \in W$. $T(x) \in W$ and $T^*(x) \in W$

$$(T_W)^* \circ T_W(x) \overset{\text{by (c)}}{=} (T^*)_W \circ T_W(x) = (T^*)_W \circ T(x) = (T^*) \circ T(x)$$

$$T_W \circ (T_W)^*(x) \overset{\text{by (c)}}{=} T_W \circ (T^*)_W(x) = T_W \circ T^*(x) = T \circ T^*(x)$$

So T_W is normal.

2. Sec. 6.4 Q8

8. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T*-invariant. *Hint:* Use Exercise 24 of Section 5.4.

Since $T \in \mathcal{L}(V)$ is normal. T is diagonalizable.

Since W is T-invariant. by Sec 5.4 Q24. T_W is diagonalizable.

i.e. There exists an orthonormal basis α for W. consisting of eigen vectors of T_W.

$\forall v \in \alpha$. $T_W(v) = \lambda v$.

i.e. $T(v) = \lambda v$

Since T is normal, $T^*(v) = \bar{\lambda} v \in W$, $T^*(\alpha) \subset W$

$T^*(W) = \text{span}(T^*(\alpha)) \subset W$ i.e. W is T^*-invariant.

3. Sec. 6.4 Q9

9. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$. *Hint:* Use Theorem 6.15 and Exercise 12 of Section 6.3.

T is normal. then $\|T(x)\| = \|T^*(x)\| \quad \forall x \in V$.

$$
\begin{aligned}
& x \in N(T) \\
\Leftrightarrow\ & T(x) = 0 \\
\Leftrightarrow\ & \|T(x)\| = 0 \\
\Leftrightarrow\ & \|T^*(x)\| = 0 \\
\Leftrightarrow\ & T^*(x) = 0 \\
\Leftrightarrow\ & x \in N(T^*)
\end{aligned}
$$

i.e. $N(T) = N(T^*)$

$$R(T^*) = N(T)^{\perp} = N(T^*)^{\perp} = R(T^{**}) = R(T)$$

By Sec 6.3 Q12

4. Sec. 6.4 Q10

10. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

- T is self-adjoint. then $T = T^*$ and $\langle T(x), x \rangle = \langle x, T(x) \rangle$
- $\|T(x) \pm ix\|^2 = \langle T(x) \pm ix, \ T(x) \pm ix \rangle$

$$= \|T(x)\|^2 + \langle T(x), \pm ix \rangle + \langle \pm ix, T(x) \rangle + \|\pm ix\|^2$$

$$= \|T(x)\|^2 \mp i \langle T(x), x \rangle \pm i \langle x, T(x) \rangle + \|x\|^2$$

$$= \|T(x)\|^2 + \|x\|^2$$

- $\forall x \in N(T - iI)$

$$0 \leq \|x\|^2 \leq \|T(x)\|^2 + \|x\|^2 = \|T(x) - ix\|^2 = \|(T - iI)(x)\|^2 = 0$$

So $x = 0$ i.e. $T - iI$ is 1-1 and thus invertible.

- Since $T = T^*$, $(T - iI)^* = T^* + iI^* = T - iI$

- $\langle x, \ [(T - iI)^{-1}]^* (T + iI)(y) \rangle$

$$= \langle (T - iI)^{-1}(x), \ (T + iI)(y) \rangle$$

$$= \langle (T - iI)^{-1}(x), \ (T - iI)^*(y) \rangle$$

$$= \langle (T - iI)(T - iI)^{-1}(x), \ y \rangle$$

$$= \langle x, y \rangle \quad \text{for any } x, y \in V$$

Thus $[(T - iI)^{-1}]^* \circ (T + iI) = I$

$$[(T - iI)^{-1}]^* = (T + iI)^{-1}$$

5. Sec. 6.4 Q11

11. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

(a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.
(b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$. *Hint:* Replace x by $x + y$ and then by $x + iy$, and expand the resulting inner products.
(c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.

(a)

$<T(x), x> = <x, T^*(x)> = <x, T(x)> = \overline{<T(x), x>}$

Thus $<T(x), x>$ is real.

(b) If $<T(x), x> = 0 \quad \forall x \in V$.

① $0 = <T(x+y), x+y>$

$= <T(x), x> + <T(x), y> + <T(y), x> + <T(y), y>$

$= <T(x), y> + <T(y), x>$

② $0 = <T(x+iy), x+iy>$

$= <T(x), iy> + <T(iy), x>$

$= -i<T(x), y> + i<T(y), x>$

By ① and ②, we have $<T(x), y> = 0 \quad \forall x, y \in V$

Let $y = T(x)$. Then $\|T(x)\|^2 = <T(x), T(x)> = 0$

Then $T(x) = 0, \quad \forall x \in V$ i.e. $T = T_0$

(c) If $<T(x), x>$ is real for any $x \in V$, then

$<x, T^*(x)> = <T(x), x> = \overline{<T(x), x>} = <x, T(x)>$

Let $U = T^* - T$. Then

$<x, U(x)> = <x, T^*(x)> - <x, T(x)> = 0 \quad \forall x \in V$

By (b) $U = T_0$ i.e $T = T^*$